Minimal model theory for relatively trivial log canonical pairs

Kenta Hashizume
Department of Mathematics, Graduate School of Science, Kyoto University

Background

All arguments are carried out over the complex number field $\mathbb{C}$. The goal of the minimal model theory is to find a good minimal model or a Mori fiber space of a given projective variety.

Conjecture 1 (Minimal model theory)

Let (X, Δ) be a projective log canonical pair. Then (X, Δ) has a good minimal model or a Mori fiber space.

Conjecture 2 (Finite generation of log canonical ring)

Let (X, Δ) be a projective log canonical pair such that Δ is a $\mathbb{Q}$-divisor. Then the log canonical ring

$$\bigoplus H^0(X, \mathcal{O}_X(\llcorner m(K_X + \Delta)\lrcorner))$$

is a finitely generated $\mathbb{C}$-algebra.

Definition (Good minimal model, Mori fiber space)

Let (X, Δ) be a projective log canonical pair and let $\phi : X \dashrightarrow X'$ be a birational map to a normal projective variety. We put $\Delta' = \phi_*\Delta + E$ where E is the reduced ϕ^{-1}-exceptional divisor on X'. Let $p : W \to X$ and $q : W \to X'$ be a common resolution of ϕ. We assume

- X' is $\mathbb{Q}$-factorial, and
- The divisor
$$F = p^*(K_X + \Delta) - q^*(K_{X'} + \Delta')$$
is effective and $\mathrm{Supp}\, F$ contains the strict transform of any ϕ-exceptional divisor.

Then the pair (X', Δ') is called a *good minimal model* of (X, Δ) if $K_{X'} + \Delta'$ is semi-ample.

On the other hand, the pair (X', Δ') is called a *Mori fiber space* if there is a contraction $X' \to W$ with $\dim W < \dim X'$ such that the relative Picard number $\rho(X'/W)$ is one and $-(K_{X'} + \Delta')$ is ample over W.

Remark.
The definition is slightly different from traditional one. When (X, Δ) is Kawamata log terminal, the definition and the traditional one coincide.

Known results
Conjecture 1 for (X, Δ) is proved when
· $\dim X \leq 3$, and
· (X, Δ) is Kawamata log terminal and Δ is big (cf. [3]).
Conjecture 2 is known when $\dim X \leq 4$ (cf. [4]).

Relatively trivial log canonical pair

Setting

- $\pi : X \to Z$: a contraction of normal projective varieties,
- (X, Δ): a log canonical pair such that Δ is a $\mathbb{Q}$-divisor satisfying that $K_X + \Delta \sim_{\mathbb{Q}, Z} 0$.

The above setting is a spacial case of *lc-trivial fibration*, in which information about (X, Δ) is expected to be obtained by investigating Z.

Known result 1 (Ambro's canonical bundle formula, cf. [1])
Under the above situation, suppose that (X, Δ) is Kawamata log terminal. Then there is a $\mathbb{Q}$-divisor Δ_Z on Z such that
· (Z, Δ_Z) is Kawamata log terminal, and
· $K_X + \Delta \sim_{\mathbb{Q}} \pi^*(K_Z + \Delta_Z)$.

Thanks to Ambro's canonical bundle formula, we can reduce Conjecture 1 for (X, Δ) to Conjecture 1 for (Z, Δ_Z) when (X, Δ) is Kawamata log terminal.

Remark. Today Ambro's canonical bundle formula is known only when (X, Δ) is Kawamata log terminal. Therefore we can not apply the canonical bundle formula to any log canonical pair to carry out such an inductive argument as above.

Known result 2 (cf. [2])
Under the above situation, suppose that $K_X + \Delta \sim_{\mathbb{Q}} \pi^*M$ such that
· M is big, and
· the augmented base locus $B_+(M)$ does not contain the image of any lc center of (X, Δ).
Then (X, Δ) has a good minimal model.

Main results

Theorem 1

Let $\pi : X \to Z$ be a projective surjective morphism of normal projective varieties and let (X, Δ) be a log canonical pair such that Δ is a $\mathbb{Q}$-divisor. Suppose that $K_X + \Delta \sim_{\mathbb{Q}, Z} 0$. Set $d = \dim Z$ and assume Conjecture 1 for all d-dimensional Kawamata log terminal pairs.
Then Conjecture 1 holds for the pair (X, Δ).

Theorem 2

Let $\pi : X \to Z$ be a projective surjective morphism of normal projective varieties and let (X, Δ) be a log canonical pair such that Δ is a $\mathbb{Q}$-divisor. Suppose that $K_X + \Delta \sim_{\mathbb{Q}} \pi^*D$.
If $\dim W \leq 3$ or $\dim W = 4$ and D is big, then Conjecture 1 holds for the pair (X, Δ).

Remark.

- In Theorem 1, if $K_X + \Delta$ is $\mathbb{Q}$-linearly equivalent to the pullback of a big divisor, we only need to assume Conjecture 1 for all $(d-1)$-dimensional Kawamata log terminal pairs to show Conjecture 1 for (X, Δ). Theorem 2 follows from this assertion with $d = 4$.
- In the proof of Theorem 1, the morphism $(X, \Delta) \to Z$ is replaced with a birational model $(X', \Delta_{X'}) \to Z'$ several times, and in this process we need the assumption that Δ is a $\mathbb{Q}$-divisor. So we can not generalize easily Theorem 1 for log canonical pairs with a boundary $\mathbb{R}$-divisor.

Theorem 3

Let (X, Δ) be a projective log canonical pair such that Δ is a $\mathbb{Q}$-divisor and the log Kodaira dimension $\kappa(X, K_X + \Delta)$ is nonnegative. Let F be the general fiber of the Iitaka fibration and (F, Δ_F) be the restriction of (X, Δ) to F. Suppose that (F, Δ_F) has a good minimal model.
If (X, Δ) is Kawamata log terminal or $\kappa(X, K_X + \Delta) \leq 4$, then (X, Δ) has a good minimal model.

Sketch of the proof.

- Take the Iitaka fibration $f : X \dashrightarrow W$. By taking an appropriate modification, we can assume that f is a morphism, W is smooth and all fibers of f have the same dimension.
- Run the $(K_X + \Delta)$-MMP over W with scaling of an ample divisor, which is a special kind of the minimal model program. Then we get a resulting good minimal model $(X', \Delta_{X'})$ over W such that $K_{X'} + \Delta_{X'} \sim_{\mathbb{Q}, W} 0$.
- By construction $K_{X'} + \Delta_{X'}$ is $\mathbb{Q}$-linearly equivalent to the pullback of a big divisor on W. Then $(X', \Delta_{X'})$ has a good minimal model by Theorem 2, and so does (X, Δ).

Corollary 1

Let (X, Δ) be a projective log canonical pair such that $\dim X = 5$ and Δ is a $\mathbb{Q}$-divisor. If $K_X + \Delta$ is not big, then the log canonical ring

$$\bigoplus H^0(X, \mathcal{O}_X(\llcorner m(K_X + \Delta)\lrcorner))$$

is a finitely generated $\mathbb{C}$-algebra.

Sketch of the proof.

- We may assume that $\kappa(X, K_X + \Delta) > 0$.
- Let $X \dashrightarrow W$ be the Iitaka fibration. Then we have $1 \leq \dim W \leq 4$.
- If $\dim W = 1$, we can find a $\mathbb{Q}$-divisor B on W and $k > 0$ such that
$$\bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(mk(K_X + \Delta))) \simeq \bigoplus_{m \geq 0} H^0(W, \mathcal{O}_W(mkB)).$$
Thus the log canonical ring is finitely generated.
- If $\dim W \geq 2$, then dimension of the general fiber of the Iitaka fibration is not greater than three. Therefore finite generation of the log canonical ring follows from Theorem 3.

References.

[1] F. Ambro, The moduli b-divisor of an lc trivial fibration, Compos. Math. **141** (2005), no. 2, 385–403.

[2] C. Birkar, Z. Hu, Log canonical pairs with good augmented base loci, Compos. Math. **150** (2014), no. 4, 579–592.

[3] C. Birkar, P. Cascini, C. D. Hacon, J. McKernan, Existence of minimal models for varieties of log general type, J. Amer. Math. Soc. **23** (2010), no. 2, 405–468.

[4] O. Fujino, Finite generation of the log canonical ring in dimension four, Kyoto J. Math. **50** (2010), no. 4, 671–684.